Acta Phys. Hung. B 20/1 (2004) 000–000

QUANTUM
ELECTRONICS

Witnessing Entanglement with Second-Order Interference and Stokes Parameters

Magdalena Stobińska[1] and Krzysztof Wódkiewicz[1,2]

[1] Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, 00-681 Warszawa, Poland

[2] Department of Physics and Astronomy, University of New Mexico, Albuquerque, NM 87131-1156, USA

Received 1 August 2006

Abstract. We propose to use Stokes parameter as an entanglement witness for correlated EPR mixed states of light. Such states can be generated with a beam splitter acting on two mixed squeezed states of light. Stokes witness operators are closely related to the Hanbury-Brown and Twiss interference and can be used to test entanglement in balanced homodyne experiments involving fluctuations of quantum quadratures of the electric field.

Keywords: Entanglement witness, Stokes parameters.
PACS: 42.50.-p

1. Introduction

In the famous paper questioning the reality of quantum mechanics, Einstein, Podolsky and Rosen [1] introduced a quantum state with nonclassical correlations between two spatially separated massive particles. In its original formulation this EPR correlated state was presented in the following general form

$$\Psi(x_1,x_2)=\sum_{n=1}^{\infty}\psi_n(x_1)u_n(x_2). \tag{1}$$

In this remarkable formula we recognize the Schmidt decomposition of an entangled state. The EPR understanding of the probabilistic interpretation of quantum mechanics has been criticized by N. Bohr [2]. This famous debate about the foundations of quantum theory and the meaning of entanglement has drawn the attention of many physicists.

1589-9535/04/ $ 20.00

As an example of a correlated state given by Eq.(1), EPR have used the following wave function

$$\Psi(x_1, x_2) \simeq \delta(x_1 - x_2). \tag{2}$$

This wave function reflects the presence of perfect correlations between the coordinates of the two particles. If the first particle is at x_1, the other one is at x_1 with certainty. Similarly, there exists a perfect correlation in the momenta of the two particles

$$\widetilde{\Psi}(p_2, p_2) \simeq \delta(p_1 + p_2). \tag{3}$$

The most important tools used in tests of quantum entanglement have been based on various Bell inequalities. These inequalities became a driving force behind a very active experimental branch of a quantum optics [3, 4, 5, 6]. Most of the experimental tests have been limited to entangled states of two qubits. For entangled systems described by continuous variables like the EPR wave function, Bell inequalities of some kind are violated as it has been demonstrated recently [7].

The entangled states defined by the Schmidt decomposition are systems described by wave functions, i.e., are pure quantum states. For a quantum system described by a mixed statistical operator ρ, it turns out that the concept of quantum entanglement, as defined by (1), has to be generalized. In the general case of a density operator, rather than a wave function, one uses the definition of quantum separability introduced by Werner [8]. A general quantum density operator of a two-party system is separable if it is a convex sum of product states,

$$\rho = \sum_k p_k \, \rho_1^{(k)} \otimes \rho_2^{(k)} \quad \text{with} \quad \sum_k p_k = 1 \quad \text{and} \quad p_k > 0\,, \tag{4}$$

where $\rho_1^{(k)}$ and $\rho_2^{(k)}$ are statistical operators of the two subsystems in question. If this criterion is not satisfied the state is called non-separable or simply entangled.

The problem of establishing which mixed state is separable and which is not, is much more complex, and the question of experimental tests for such systems is still an open problem. Therefore, within the last decade a notion of an entanglement witness has generated a lot of interest as a experimentally feasible way of entanglement test for mixed states. The idea relies on constructing an operator which mean value is positive for separable and negative for entangled states

$$\langle \mathcal{W} \rangle = \begin{cases} \geq 0, & \rho \text{ is separable,} \\ < 0, & \rho \text{ is entangled.} \end{cases}$$

Such an operator has been discussed first for discrete variables [9]. A general method for the experimental detection of entanglement for qubits is discussed in [10]. The other experimentally feasible method relies on violating the entropy inequalities [11]. Another entanglement witness for continuous variables, based on the measurement of the fluctuations of photocurrent difference between two correlated modes of detected light, has been introduced in [12].

A method relying on the Hanbury-Brown and Twiss interference, in which one can introduce an entanglement witness operator $\mathcal{W}^{(HBT)}$ has been presented in [13].

In this paper we explicitly construct an operator which witnesses continuous variable entanglement of the EPR state. This operator can be expressed in terms of quantum uncertainties of the Stokes parameter. We show that the measurement of witness operator is equivalent to homodyne detection of quantum correlations and fluctuations of the electric field quadratures. If the value of the quantum uncertainty is below the shot noise the state is entangled. Our work focuses on general mixed gaussian states, involving as an example mixed EPR states.

2. Thermal mixed squeezed states

Let us start our discussion with a general one-mode mixed gaussian state. A single mode electric field operator, oscillating with frequency ω, can be written in the following form

$$E = E_0(ae^{-i\omega t} + a^\dagger e^{i\omega t}) = \sqrt{2}E_0(X_1 \cos\omega t + X_2 \sin\omega t), \tag{5}$$

where $X_1 = \frac{a+a^\dagger}{\sqrt{2}}$ and $X_2 = \frac{a-a^\dagger}{\sqrt{2}i}$ are amplitude and phase hermitian quadratures of the electric field.

A one-mode gaussian state is completely characterized by the elements of its covariance matrix given by the following second-order moments

$$\langle a^\dagger a\rangle = \bar{n},\ \langle a^2\rangle = -m, \tag{6}$$

where $\bar{n}$ is a mean number of photons in the mode, and m is a squeezing parameter, which we assume for simplicity to be real. We have assumed that $\langle a\rangle = \langle a^\dagger\rangle = 0$, which can always be arranged with a suitable unitary shift of a and $a^\dagger$.

Knowing those two parameters it is easy to construct a positive density operator for the state

$$\rho_a = \frac{e^{-N\,a^\dagger a - M^* a^2 - M\,a^{\dagger 2}}}{Z}, \tag{7}$$

where N and M are known functions of $\bar{n}$ and m [14]. In this formula Z is the partition function. This mixed squeezed state leads to the following uncertainty of electric field quadratures

$$\Delta X_1 = \sqrt{\bar{n} + \frac{1}{2} - m},\ \Delta X_2 = \sqrt{\bar{n} + \frac{1}{2} + m}, \tag{8}$$

and the Heisenberg uncertainty relation is

$$\Delta X_1 \Delta X_2 = \sqrt{\left(\bar{n} + \frac{1}{2}\right)^2 - m^2}, \tag{9}$$

which imposes the following condition for the squeezing parameter

$$0 \leq |m| \leq \sqrt{\bar{n}(\bar{n}+1)}. \tag{10}$$

It is clear that this condition guarantees that $\rho_a \geq 0$. The notion that m is the squeezing parameter of quantum quadratures is obvious from these relations. Squeezing means in this case fluctuations below the vacuum noise, i.e.,

$$\bar{n} < |m|. \tag{11}$$

For $|m| = \sqrt{\bar{n}(\bar{n}+1)}$, this state reduces to the single mode pure state.

3. Entangling photons

One of the simplest way to entangle photons is to use a beam splitter. The action of the beam splitter is a nonlocal operation that may, in general, produce correlated photons. The use of a beam splitter to entangle squeezed light has been implemented in several experiments [15, 16]. This process is described as a 50/50 beam splitter (BS) transformation of a product state into a correlated state

$$BS: \;\; \rho_a \otimes \rho_b \mapsto \rho_{cd}. \tag{12}$$

The operators $\rho_{a,b}$ are the density operators of the single mode mixed squeezed states with equal number of photons but opposite squeezing phases

$$\begin{aligned} \langle a^\dagger a \rangle &= \langle b^\dagger b \rangle = \bar{n}, \\ \langle a^2 \rangle &= -\langle b^2 \rangle = -m. \end{aligned} \tag{13}$$

The relation between input and output modes operators on a 50/50 beam splitter have the well known form

$$c = \frac{a+b}{\sqrt{2}}, \; d = \frac{a-b}{\sqrt{2}}. \tag{14}$$

The two output beams form a correlated squeezed mixed EPR state with equal mean number of photons in each mode

$$\langle c^\dagger c \rangle = \langle d^\dagger d \rangle = \bar{n} \tag{15}$$

and with the following correlation parameter

$$\langle cd \rangle = \frac{\langle a^2 \rangle - \langle b^2 \rangle}{2} = -m. \tag{16}$$

It is clear that the beam splitter correlates the two beams

$$BS: \;\; (\bar{n}, m)_a \otimes (\bar{n}, -m)_b \mapsto (\bar{n}, m)_{cd}. \tag{17}$$

For $|m| = \sqrt{\bar{n}(\bar{n}+1)}$ this state reduces to the well known pure two-mode squeezed state, generated in a process of nondegenerate optical parametric amplification (NOPA)

$$|\text{NOPA}\rangle = \sum_{n=0}^{\infty} \sqrt{p_n}|n,n\rangle, \tag{18}$$

where $p_n = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}}$ is a Bose-Einstein photon distribution. This state is the closest experimental realization, of the original EPR state. In the limit of infinite intensity, $\bar{n} \to \infty$, this state with $p_n \sim 1$ becomes the EPR state (2)

$$\lim_{\bar{n}\to\infty} \langle x_1, x_2|\text{NOPA}\rangle = \sum_{n=0}^{\infty} \langle x_1, x_2|n,n\rangle \sim \delta(x_1 - x_2). \tag{19}$$

The separability condition for the two mode mixed state has been addressed in several publications. In this paper we follow the results obtained in [17, 18]. Figure 1 shows the results computed for the general mixed EPR state.

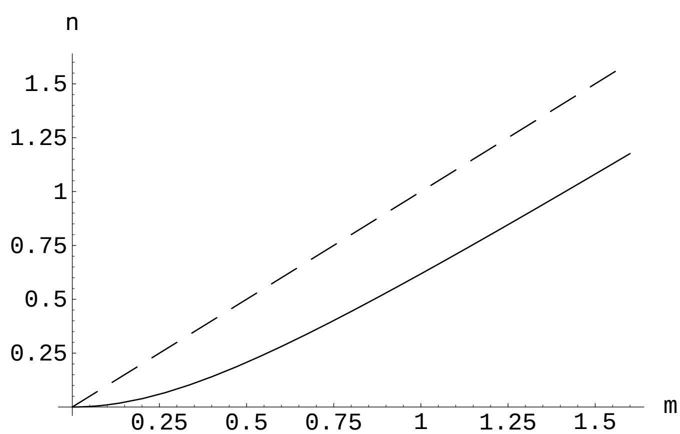

Fig. 1. The values on and above the curve $\bar{n} = \frac{1}{2}\left(-1+\sqrt{1+4m^2}\right)$ correspond to the physical states (with a positive density operator) - EPR mixed states. The states between the curve and the dashed line $\bar{n} = m$ are entangled. The EPR states above the dashed line are classically correlated, so separable.

4. Witnessing entanglement

In reference [13] it has been shown that the entanglement witness operator, related to the second-order Hanbury-Brown and Twiss interference, is given by the following

hermitian operator

$$\mathcal{W}^{(HBT)} = \frac{1}{2} - \frac{2a^\dagger b^\dagger ab + {b^\dagger}^2 a^2 + {a^\dagger}^2 b^2}{\langle : (I_a + I_b)^2 : \rangle}. \tag{20}$$

The mean value of this operator allows to witness the entanglement in the outcoming modes c and d. Using Eq. (14) we obtain

$$\mathrm{Tr}\{\mathcal{W}^{(HBT)}\rho\} = \frac{\bar{n}^2 - |m|^2}{2(3\bar{n}^2 + |m|^2)} = \begin{cases} \geq 0, & \text{separable,} \\ < 0, & \text{entangled.} \end{cases}$$

5. Hong-Ou-Mandel interference

The entanglement witness which has been introduced in the last section is experimentally feasible quantity. It is directly related to the second-order visibility $v^{(2)}$ observed in the Hong-Ou-Mandel time resolved interference.

The joint coincidence of detecting photons from mode c and d at delayed times t and $t+\tau$ can be obtained directly from the following second-order temporal coherence function

$$G^{(2)}(t, t+\tau) = \left\langle E_c^{(-)}(t)\, E_d^{(-)}(t+\tau)\, E_d^{(+)}(t+\tau)\, E_c^{(+)}(t) \right\rangle. \tag{21}$$

Assuming the random phases in the output state and random phases typical for a stationary stochastic phase diffusion model in mode functions, we achieve the probability of the joint detection depending only on the delay [19]

$$p(\tau) = 1 - v^{(2)} \exp\left\{ - \left(\frac{\tau}{\tau_c} \right)^2 \right\}, \tag{22}$$

where τ_c is a coherence time. For the EPR correlated state, the visibility of interference fringes is equal to

$$v^{(2)} = \frac{\bar{n}^2 + |m|^2}{3\bar{n}^2 + |m|^2}. \tag{23}$$

The visibility is directly related to entanglement witness

$$\langle \mathcal{W}^{(HBT)} \rangle = \frac{1}{2} - v^{(2)}. \tag{24}$$

If the EPR state is separable, $\bar{n} \geq |m|$, the visibility is $v^{(2)} \leq \frac{1}{2}$ and the witness (24) is positive. For entangled states $v^{(2)} > \frac{1}{2}$, the witness takes a negative value.

6. Stokes Parameters

The Stokes operators for the outgoing modes c and d are given by the following expressions

$$S_x = \frac{c^\dagger d + d^\dagger c}{2},\ S_z = \frac{c^\dagger c - d^\dagger d}{2}, \tag{25}$$

$$S_y = \frac{c^\dagger d - d^\dagger c}{2i},\ S_0 = \frac{c^\dagger c + d^\dagger d}{2}. \tag{26}$$

These definitions of the Stokes operators are known as the Schwinger representation of the angular momentum operators in terms of annihilation operators of two harmonic oscillators. These operators provide a very useful and efficient tool in the theoretical and experimental description of polarization entangled states [20]. In this paper we will use the Stokes parameters for continuous variable entangled squeezed states.

The Stokes operators form a set of noncommuting operators

$$[S_x, S_y] = iS_z \tag{27}$$

and their joint measurement is limited by the Heisenberg uncertainty relation

$$(\Delta S_y)^2(\Delta S_z)^2 \geq |\langle S_x\rangle|^2, \tag{28}$$

which is equivalent to a positive density matrix condition $\bar{n}(\bar{n}+1) \geq |m|^2$.

The entanglement witness (20) can also be expressed in terms of the normally ordered Stokes parameters

$$\langle \mathcal{W}^{(HBT)}\rangle = \frac{1}{2} - \frac{\langle : S_x^2 :\rangle}{\langle : S_x^2 :\rangle + \langle : S_y^2 :\rangle + \langle : S_z^2 :\rangle} \tag{29}$$

$$= \frac{1}{2} - \frac{\langle S_x^2\rangle - \frac{1}{2}\langle S_0\rangle}{\langle S_x^2\rangle + \langle S_y^2\rangle + \langle S_z^2\rangle - \frac{3}{2}\langle S_0\rangle}. \tag{30}$$

The mean value of the witness takes a negative value if any of the following inequalities is obeyed

$$\langle : S_x^2 :\rangle > \langle : S_y^2 :\rangle + \langle : S_z^2 :\rangle, \tag{31}$$

$$\langle S_x^2\rangle > \langle S_y^2\rangle + \langle S_z^2\rangle - \frac{1}{2}\langle S_0\rangle. \tag{32}$$

As we shall see below, the measurement of the uncertainty of a single Stokes parameter can also constitute an entanglement witness. The measurement of ΔS_z seems to be the best choice for an experimental realization. This measurement can be done using balanced homodyne detection with a strong local oscillator beam $\alpha = |\alpha|e^{i\varphi}$. In Fig. 2, we have depicted an experimental scheme of such a measurement. The two incoming modes a and b (13) are correlated at a 50/50 beam splitter. The detectors measure intensities of the new displaced modes $\tilde{c}$ and $\tilde{d}$. The Stokes parameter S_z for the detected outgoing modes $\tilde{c}$ and $\tilde{d}$ is given by

$$S_z = \frac{\tilde{c}^\dagger \tilde{c} - \tilde{d}^\dagger \tilde{d}}{2}, \tag{33}$$

where the modes $\tilde{c}$ and $\tilde{d}$ are displaced by the strong pump

$$\tilde{c} = c + \alpha,\ \tilde{d} = d + \alpha. \tag{34}$$

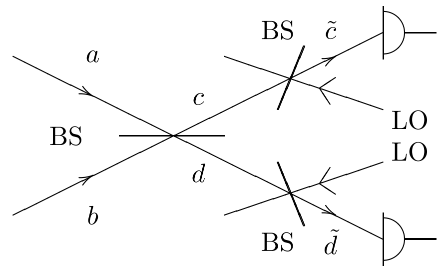

Fig. 2. Balanced homodyne detection. The outgoing modes, c and d given by (15) and (16), are combined with a strong local oscillator beam at a 50/50 beam splitter.

Defining rotated quadratures as

$$X_c(\varphi) = \frac{c\, e^{-i\varphi} + c^\dagger e^{i\varphi}}{\sqrt{2}},\ X_d(\varphi) = \frac{d\, e^{-i\varphi} + d^\dagger e^{i\varphi}}{\sqrt{2}}, \tag{35}$$

the uncertainty of Stokes parameter S_z is equal to

$$(\Delta S_z)^2 = \frac{\bar{n}(\bar{n}+1) - m^2}{2} + \frac{|\alpha|^2}{2} \left\langle (X_c(\varphi) - X_d(\varphi))^2 \right\rangle . \tag{36}$$

The above formula simplifies to

$$(\Delta S_z)^2 = \frac{\bar{n}(\bar{n}+1) - m^2}{2} + \frac{|\alpha|^2}{2} \left(1 + 2(\bar{n} + m\cos 2\varphi)\right), \tag{37}$$

where $\langle X_c(\varphi) X_d(\varphi)\rangle = m\cos 2\varphi$ is the correlation between the two quadratures. Choosing $\varphi = \frac{\pi}{2}$ and keeping quadratic terms in α only, the formula (37) reduces to

$$(\Delta S_z)^2 \sim \frac{|\alpha|^2}{2}(1 + 2(\bar{n} - m)). \tag{38}$$

The above expression shows that if the EPR state is entangled, so if $m > \bar{n}$ the uncertainty of S_z is below the shot noise

$$(\Delta S_z)^2 < \frac{|\alpha|^2}{2}. \tag{39}$$

In a similar way other Stokes parameters can also serve as an entanglement witness.

Evaluating the variance of the Stokes parameter S_x

$$S_x = \frac{\tilde{c}^\dagger \tilde{d} + \tilde{d}^\dagger \tilde{c}}{2}, \tag{40}$$

with $\langle S_x\rangle = |\alpha|^2$ we get

$$(\Delta S_x)^2 = \frac{\bar{n}(\bar{n}+1) + m^2}{2} + \frac{|\alpha|^2}{2} \left\langle (X_c(\varphi) + X_d(\varphi))^2 \right\rangle . \tag{41}$$

In the strong field limit this formula simplifies to

$$(\Delta S_x)^2 \sim \frac{|\alpha|^2}{2}\left(1+2(\bar{n}-m\cos 2\varphi)\right). \tag{42}$$

Choosing $\varphi = 0$ and keeping quadratic terms in α only, the formula (42) reduces to formula (38). The outcoming state is entangled if $(\Delta S_x)^2$ is smaller than for a shot noise. Similarly for S_y

$$S_y = \frac{\tilde{c}^\dagger \tilde{d} - \tilde{d}^\dagger \tilde{c}}{2}. \tag{43}$$

Its mean value $\langle S_y \rangle = 0$ and the variance is equal to

$$(\Delta S_y)^2 = \frac{n(n+1)+m^2}{2} + \frac{|\alpha|^2}{2}\left\langle \left(X_c\left(\varphi+\frac{\pi}{2}\right) - X_d\left(\varphi+\frac{\pi}{2}\right)\right)^2\right\rangle. \tag{44}$$

This formula reduces to formula (42) giving the same condition for non-separability of outcoming EPR mixed state.

7. Conclusions

We have proposed a measurement of the uncertainties of Stokes parameter $\bar{S} = (S_x, S_y, S_z)$ as an entanglement witness for a general class of EPR mixed correlated states. This method seems to be relatively easy to implement experimentally. If the quantum fluctuations of the Stokes paramaters is below the shot noise (the value obtained for a vacuum state), the state is entangled i.e., non-separable.

Acknowledgements

This work was partially supported by a MEN Grant No. 1 PO3B 137 30.

References

1. A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777, (1935).
2. N. Bohr, Phys. Rev. **48**, 696, (1935).
3. A. Aspect, Ph. Grangier, and G. Roger Phys. Rev. Lett. **47**, 460 (1982).
4. G. Weihs, T. Jennewein, Ch. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **81**, 5039 (1998).
5. W. Tittel, J. Brendel, B. Gisin, T. Herzog, H. Zbinden, and N. Gisin, Phys. Rev. A **57**, 3229 (1998).
6. R. Garcia-Patron, J. Fiurasek, N. Cerf, J. Wenger, R. Tualle-Brouri, and Ph. Grangier Phys. Rev. Lett. **93**, 130409 (2004).
7. K. Banaszek and K. Wódkiewicz, Phys. Rev. A **58**, 4345 (1998).
8. R. F. Werner, Phys. Rev. **40**, 4277 (1989).

9. M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Phys. Rev. A **62**, 052310 (2000).
10. O. Guhne, P. Hyllus, D. Bruss, A. Ekert, M. Lewenstein, C. Macchiavello, and A. Sanpera, Phys. REv. A **66**, 062305 (2002).
11. P. Horodecki, Phys. Rev. A **68**, 052101, (2003).
12. M. Paris, Phys. Rev. A **59**, 1615 (1999).
13. M. Stobińska and K. Wódkiewicz, Phys. Rev. A **71**, 032204 (2005).
14. See for example: C. W. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods With Applications to Quantum Optics* (Springer-Verlag, Berlin, 2000).
15. O. Glöckl, S. Lorenz, C. Marquardt, J. Heersink, M. Brownnutt, Ch. Silberhorn, Q. Pan, P. van Loock, N. Korolkova, and G. Leuchs, Phys. Rev. A **68**, 012319 (2003).
16. E. H. Huntington, G. N. Milford, C. Robilliard, T. C. Ralph, O. Glöckl, U. L. Andersen, S. Lorenz, and G. Leuchs, Phys. Rev. A **71**, 041802(R), (2005). **65**, 054303 (2002).
17. B. G. Englert and K. Wódkiewicz, Int. J. of Q. Info. **1**, 153 (2003).
18. B. G. Englert and K. Wódkiewicz, Phys. Rev. A **65**, 054303 (2002).
19. M. Stobińska and K. Wódkiewicz, IJMP B **20**, 1504 (2006).
20. J. Heersink, T. Gaber, S. Lorenz, O. Glöckl, N. Korolkova, and G. Leuchs, Phys. Rev. A **68**, 013815 (2003).